ON THE PROBLEM OF ENCOUNTER IN SECOND-ORDER SYSTEMS WITH IMPULSIVE AND NONPARALLEL CONTROLS

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Two problems on the minimax of the time until encounter of the objects are considered. The control vectors of the pursuer and pursued object are assumed to be of fixed direction and nonparallel. Each of the vectors is subject to an "impulsive" restriction. The paper is related to [1, 2] in subject matter. The results of computations carried out by Iu. A. Kraiushkin are cited.

1. Let the relative motion of the pursuer $P(y_1, y_2)$ and pursued object $E(z_1, z_2)$ in the variables $x_1 = y_1 - z_1$, $x_2 = y_2 - z_2$ be described by the equations

$$x_1^{\cdot} = x_2 + v \cos \varphi = x_2 + v\gamma_1$$
$$x_2^{\cdot} = -x_1 + v \sin \varphi + u = -x_1 + v\gamma_2 + u \tag{1.1}$$

and let the controls u and v be subject to the restrictions

$$\mu(\tau) = M - \int_0^\tau |u| \, d\tau \geqslant 0, \qquad M = \text{const} > 0$$
$$\nu(\tau) = N - \int_0^\tau |v| \, d\tau \geqslant 0, \qquad N = \text{const} > 0 \tag{1.2}$$

The control $v(\tau)$ is formed in accordance with the phase vector (x_1, x_2, μ, ν) of the system; moreover, the pursuer also knows the control $v(\tau)$, i.e. the objects are discriminated,

$$v(\tau) = v[x_1(\tau), x_2(\tau), \mu(\tau), \nu(\tau)]$$
$$u(\tau) = u[x_1(\tau), \ldots, \nu(\tau), v(\tau)]$$

Restrictions (1.2) admit of controls in the form of impulsive δ-functions $u = \mu_1\delta$, $v = \nu_1\delta$, so that we shall assume that the variables in this case vary according to the formulas

$$x_1(\tau) = x_1(\tau - 0) + \nu_1\gamma_1, \qquad x_2(\tau) = x_2(\tau - 0) + \nu_1\gamma_2 + \mu_1$$
$$\mu(\tau) = \mu(\tau - 0) - |\mu_1|, \qquad \nu(\tau) = \nu(\tau - 0) - |\nu_1| \tag{1.3}$$

We define a "permissible trajectory" $x_1(\tau)$, $x_2(\tau)$, $\mu(\tau)$, $\nu(\tau)$ as a trajectory with a finite number of jumps which is everywhere continuous on the right and satisfies Eqs. (1.1) together with the Eqs. $\mu^{\cdot} = -|u|$, $\nu^{\cdot} = -|v|$ almost everywhere. A pair of equations realizing a permissible trajectory will be called "mutually permissible". It is to this class that we confine our solution of the problem. We shall call the set of quantities x_1, x_2, μ, ν the "phase vector" and denote it by z. The possibility of jumps requires refinement of the notion of "encounter".

Definition 1.1. Let (x_1, x_2, μ, ν) be the initial point or the left-hand limits of the trajectory as $t \to T - 0$, and let $v = \nu_1(x_1, \ldots, \nu)\delta$ be the impulsive control of the pursued object. If there exist an impulsive control $u = \mu_1\delta$ and a number $0 \leqslant \lambda \leqslant 1$ which satisfy the equations

$$x_1 + \lambda\nu_1\gamma_1 = 0, \qquad x_2 + \lambda\nu_1\gamma_2 + \mu_1 = 0$$

then the set $(x_1, x_2, \mu, \nu, \nu_1)$ which includes the impulse ν_1 in addition to the phase vector will be called the "encounter" and the quantity T the "instant of encounter". We note that an encounter is realized if and only if a closed segment with the origin (x_1, x_2) and the components $(\nu_1\gamma_1, \nu_1\gamma_2)$ has at least one point in common with the closed segment $x_2 = 0$, $|x_1| \leqslant \mu$.

We introduce the following new variables:

$$\eta_1 = x_1 / \mu, \qquad \eta_2 = x_2 / \mu, \qquad \alpha = \nu / \mu$$
$$u_1 = u / \mu, \qquad v_1 = v / \mu$$

For finite u and v expressions (1.1), (1.2) give us the equations

$$\eta_1^{\cdot} = \eta_2 + \eta_1 |u_1| + v_1\gamma_1, \qquad \eta_2^{\cdot} = -\eta_1 + \eta_2 |u_1| + u_1 + v_1\gamma_2$$
$$\alpha^{\cdot} = -|v_1| + \alpha |u_1| \tag{1.4}$$

Equations (1.3) become

$$\eta_1'(\tau) = \lambda [\eta_1 + \nu_1\gamma_1 / \mu], \qquad \eta_2'(\tau) = \mp 1 \pm \lambda (1 \pm \nu_1\gamma_2 / \mu) \tag{1.5}$$
$$\alpha'(\tau) = \lambda [\alpha - |\nu_1| / \mu], \qquad \lambda = \mu / (\mu - |\mu_1|)$$

Here the upper combination corresponds to $\mu_1 > 0$ and the lower to $\mu_1 < 0$; the quantities η_1, η_2, α are the left-hand limits or initial point.

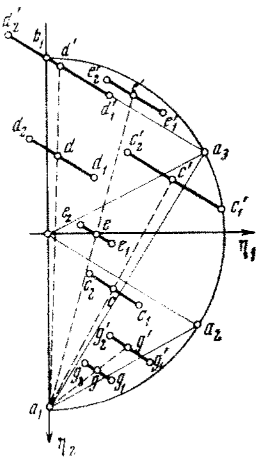

Fig. 1

We propose to solve the problem of finding a pair of strategies u°, v° which yield the saddle point of the cost $T(u, v)$ of the game. As will be shown below, saddle strategies do not always exist. In certain cases it is possible only to specify a strategy u° and a sequence of strategies $v^\circ(\varepsilon)$ for which

$$\min_u \sup_v T = \sup_v \min_u T = \lim_{\varepsilon \to 0} \sup_{v_\varepsilon} \min_u T = T^\circ$$

where the limit function $\lim v_\varepsilon$ results in encounter at $T_1 < T^\circ$ as $\varepsilon \to 0$. If there exists a v° such that encounter does not occur for any u for $v = v^\circ$, this v° is also "optimal".

2. Let us direct the axis η_2 downwards and consider the unit semicircle (a_1, a_2, b_1) in the plane $\eta_1\eta_2$. Next, let us represent the several phase vectors (η_1, η_2, α) by means of the point triplets (g_2, g, g_1), (c_2, c, c_1), (e_1, e, e_1) (see Fig. 1). Each of the triplets, e.g. (g_2, g, g_1) corresponds to a point $g(\eta_1, \eta_2)$ and an arrow (g_2, g_1) of length 2α parallel to the radius $(0, a_2)$ forming the angle φ with the axis η_1.

Formulas (1.5) indicate that the impulsive control $u = \mu_1\delta < 0$ transforms the vector (g_2, g, g_1) into the vector (g_2', g', g_1') by way of a similarity transformation along the straight lines (a_1, g_1), (a_1, g_2). By virtue of the linearity of the problem, the optimal control changes sign with a change in the signs of η_1, η_2, while the optimal time remains unchanged. We shall therefore confine our attention to the domain $\eta_1 > 0$ and stipulate that φ is acute.

The same problem for $\varphi = \pi / 2$ is solved in [2]; the control $u^\bullet = -\mu_1^\circ\delta$. The control in this problem is impulsive and must be chosen on the basis of the equation

$$R' = \sqrt{\eta_1'^2 + \eta_2'^2} = 1 - \alpha'$$

where η_1', η_2', α' are replaced by the right sides of (1.5). On the other hand, if the latter equation is fulfilled for $\mu_1^\circ = 0$, the u° $(v) < 0$ must be chosen from the condition $R_1^{\cdot} + \alpha^{\cdot} = 0$.

The control v° can be assumed to equal zero or to any positive value as long as the arrow ends g_2 and g_1 remain inside the unit semicircle. On the other hand, if one of them, let us say g_2, lies outside the semicircle, the control $v^\circ = -\nu\delta < 0$ takes the point out of the semicurcle and encounter cannot occur at any subsequent time. Upon realization of an optimal control u° this does not occur and encounter takes place after the optimal time $T^\circ = 2 \operatorname{arc} \operatorname{tg} [\eta_1/(1 - \eta_2 - \alpha)]$. On the other hand, if the chosen impulse μ_1 is smaller than $\mu_1^\bullet$, the point g_2 leaves the semicircle prior to encounter, and encounter does not ensue.

Let us begin our analysis of the general case and choose the smallest possible impulsive control $u^\bullet = \mu_1^\circ\delta < 0$ for which none of the points $g_1, g_2, c_1, c_2\ldots$ can leave the semicircle for a subsequent realization of $u = v = 0$. Let the point g_1 lie to the right of the straight line (a_1a_2) and let $R + \alpha = \sqrt{\eta_1^2 + \eta_2^2} + \alpha > 1$, i.e. let the phase vector lie in the domain $D_1^\bullet$ defined by the inequalities

$$D_1^\bullet\ [\psi_1 = (\eta_1 + \alpha\gamma_1)\cos\theta - \sin\theta\,(1 - \eta_2 - \alpha\gamma_2) > 0, \quad \theta = \varphi / 2 + \pi / 4,$$
$$R + \alpha > 1]$$

2.1. If the phase vector lies in the domain D_1° and the control $v^\bullet = 0$ as long as the point g_1 is inside the semicircle, and if $v^\bullet = \nu\delta > 0$ when it leaves the semicircle, the encounter cannot occur for any control u.

Proof. The derivative $(R + \alpha)^{\cdot}$ satisfies the following estimate in the domain D_1°:

$$(R + \alpha)^{\cdot} = (R + \alpha)\,|u_1| + (\eta_2 / R)\,u_1 \geqslant (R + \alpha - 1)\,|u_1| \geqslant 0$$

This means that $R + \alpha$ does not decrease and that the point g_1 will lie outside the semicircle at the instant of emergence of the point g onto the radius $(0, a_2)$. At this instant or somewhat earlier it is possible to realize $v = \nu\delta > 0$ so that the point g will lie outside the semicircle which is the attainability domain of zero [3] until emerging onto the radius $(0,\ a_2)$. The proof is now complete.

2.2. Following 2.1, let us choose a control $u^\bullet = \mu_1^\circ\delta < 0$ on the basis of the condition $R' + \alpha' = 1$ on the surface $R + \alpha = 1$ in the domain $D_1\ [\psi_1 > 0,\ R + \alpha \leqslant 1]$ bounding the domain $D_1^\bullet$. We choose this control in the form

$$\lambda_1^\circ = 2\,(1 - \eta_2 - \alpha)\,/\,[\eta_1^2 + (1 - \eta_2)^2 - \alpha^2] \qquad (2.1)$$
$$\lambda^\bullet = \mu\,/\,(\mu - |\,\mu_1^\circ\,|)$$

If an impulsive control $v = \nu_1\delta$ has been realized for the vector (η_1, η_2, α) in the domain D_1 and if the point

$$(\eta_1 + (\nu_1 / \mu)\gamma_1,\quad \eta_2 + (\nu_1 / \mu)\gamma_2,\ \alpha - |\,\nu_1\,| / \mu)$$

has remained in the domain D_1, these values must be substituted in place of η_1, η_2, α in formula (2.1). This note is also valid for all the subsequent expressions for λ_2°, $\lambda_3^\bullet$, λ_4°. If $R + \alpha = 1$, the final control u_1° obtained from the condition $(R + \alpha)^{\cdot} = 0$ is realized in accordance with the condition

$$u_1^\circ = (1 - \eta_2)^{-1}\,[v_1\,(\eta_1\gamma_1 + \eta_2\gamma_2) - |\,v_1\,|\,R] \qquad (2.2)$$

2.3. Let us choose the control $u^\circ = \mu_1^\circ\delta$ on the basis of the condition $(\eta_1' + \alpha'\gamma_1)^2 + (\eta_2' + \alpha'\gamma_2)^2 = 1$ of arrival of the point on the arc (a_2a_3) for the vector (c_2, c, c_1) corresponding to the position of the point c_1 $(\eta_1 + \alpha\gamma_1, \eta_2 + \alpha\gamma_2)$ between the straight lines (a_1a_3) and (a_1a_2) in the domain

$$D_2\begin{bmatrix}\psi_1 \leqslant 0,\ \psi_2 = (\eta_1 + \alpha\gamma_1)\gamma_1 - \gamma_2(1 - \eta_2 - \alpha\gamma_2) > 0\\ r^2 = (\eta_1 + \alpha\gamma_1)^2 + (\eta_2 + \alpha\gamma_2)^2 \leqslant 1\end{bmatrix}$$

The quantity λ_2° is of the form

$$\lambda_2^\circ = 2(1 - \eta_2 - \alpha\gamma_2) / [(\eta_1 + \alpha\gamma_1)^2 + (1 - \eta_2 - \alpha\gamma_2)^2] \qquad (2.3)$$

If $\lambda_2^\circ = 1$, the final control $u_1^\circ < 0$ determined from the condition $(\lambda_2^\circ)^\cdot = 0$ is realized in the form

$$u_1^\circ = -(1 - \eta_2 - \alpha\gamma_2)^{-1}[(\eta_1\gamma_2 - \eta_2\gamma_1)\alpha + (\eta_1\gamma_1 - \eta_2\gamma_2 - \alpha) \times (|v_1| - v_1)] \qquad (2.4)$$

It is geometrically self-evident that the points e_1', e_2' cannot leave the semicircle during the realization of u_1° as long as the point $e(\eta_1, \eta_2, \alpha)$ lies in the domain D_2.

2.4. Passage to the vector (e_2', e', e_1') in the domain D_3' $[\psi_2 \leqslant 0,\ \psi_3 = \eta_1 - \alpha\gamma_1 > 0]$ can be hazardous for the vector (e_2, e, e_1) corresponding to the point e_1 situated no further to the left than the straight line (a_1a_3) and to the point e_2 situated to the right of the straight line $\eta_2 = 0$. The hazard consists in the possibility, that the point e_2' will reach the semicircle in the neighborhood of the point b_1.

2.5. If we set $v^\circ = 0$ in the domain

$$D_3^\circ\ [\psi_2 \leqslant 0,\ \ \psi_3 > 0,\ \ \rho^2 = \eta_1^2 + \eta_2^2 - \alpha^2 + 2\alpha\gamma_2 > 1]$$

as long as the point e_2' lies inside the semicircle and if $v^\circ = -\nu_1\delta$ once it emerges beyond the semicircle, the encounter is impossible for any control u. The proof is similar to that of Subsect. 2.1.

2.6. Let us choose the quantity λ_3° in the domain $D_3\ [\psi_2 \leqslant 0,\ \psi_3 > 0,\ \rho^2 \leqslant 1]$ on the basis of the equation $\rho'^2 = \eta_1'^2 + \eta_2'^2 - \alpha'^2 + 2^1\alpha'\gamma_2 = 1$ in the form

$$\lambda_3^\circ = 2(1 - \eta_2 - \alpha\gamma_2) / [\eta_1^2 + (1 - \eta_2)^2 - \alpha^2] \qquad (2.5)$$

For $\lambda_3^\circ = 1$ the final control u_1° can be determined from the condition $(\lambda_3^\circ)^\cdot = 0$ in the form

$$u_1^\circ = +[(\eta_1\gamma_1 + \eta_2\gamma_2)v_1 + (\alpha - \gamma_2)\,|v_1|\,] / (1 - \eta_2 - \alpha\gamma_2) \qquad (2.6)$$

2.7. Let us find a jump $\mu_1^\circ < 0$ for the vector (d_2, d, d_1) with the point d_2 lying to the left of the axis $\eta_1 = 0$ and the point d_1 lying to the left of the straight line (a_1, a_3) in the domain $D_4\ [\psi_2 \leqslant 0,\ \ \psi_3 \leqslant 0,\ \ \psi_4 = \eta_1\gamma_2 - (1 + \eta_2)\gamma_1 \leqslant 0]$

We choose this jump from the condition of arrival of the point d' at the straight line (b_1a_3) in the form

$$\lambda_4^\circ = 2(1 - \eta_2 - \eta_1 \operatorname{tg}\varphi) / (\eta_1^2 + (1 - \eta_2)^2 - \eta_1^2\gamma_2^2 \qquad (2.7)$$

For $\lambda_4^\circ = 1$ we find the control u_1° from the condition $(\lambda_4^\circ)^\cdot = 0$ in the form

$$u_1^\circ = (\eta_2 \operatorname{tg}\varphi + \eta_1) / (1 - \eta_2 + \eta_1 \operatorname{tg}\varphi) \qquad (2.8)$$

Beginning our selection of the control v°, we note that it is no longer possible to adduce the heuristic considerations concerning the minimization of μ_1° which assisted us in selecting u°. Let us simply take the control v_1° without any explanation. This choice will be justified further on in our discussion.

2.8. Let us choose a small quantity $\varepsilon(\varphi) > 0$ which vanishes as $\varphi \to \pi/2$ and let $v^\circ = 0$ to the right of the straight line $w_1 = \eta_1 / (1 - \eta_2) = \varepsilon$ in the domains D_1 and D_2.

2.9. Let us choose $v^{\circ} > 0$ on the straight line $w_1 = \varepsilon$ in the domains D_1 and D_2 on the basis of the condition $w_1^{\cdot} = 0$ and in the form

$$v_1^{\circ} = (\varepsilon\eta_1 - \eta_2) / (\gamma_1 + \varepsilon\gamma_2) \tag{2.9}$$

2.10. Let us assume that the control $v^{\circ} = -\nu\delta > 0$ is impulsive and realizes the entire safety margin in the domains D_3 and D_4.

2.11. Let us set $v^{\circ} = \pm\nu_1\delta$ in any situation which brings the points $(\eta_1 \pm \alpha\gamma_1, \eta_2 \pm \alpha\gamma_2)$ outside the semicircle.

3. Let us assume that at the initial instant the value

$$w_1^{\circ} = \eta_1^{\circ} / (1 - \eta_2^{\circ}) \geqslant \varepsilon\,(\varphi)$$

and the vector $(\eta_1^{\circ}, \eta_2^{\circ}, \alpha^{\circ})$ lie in the domain D_1 or D_2. Then the realization of the controls u°, v° chosen in accordance with Eqs. (2.1),...,(2.8) and Subsects. 2.7,... ...,2.10 ensures the realization of some motion which we shall call a "trajectory". In motion along a trajectory encounter is realized in a time T_ε which generally consists of three components, $T_\varepsilon = T_1 + T_2 + T_3$. The time $T_1 = 2 \operatorname{arc\,tg} [\eta_1 / (1 - \eta_2 - \alpha)] - \pi / 2 - \varphi$ corresponds to motion from the domain D_1 to the boundary of the domain D_2 along the circle $R' = \text{const}$. If the motion begins in the domain D_2, the time $T_1 = 0$. The time T_2 corresponds to the motion of the point in the domain D_2; $w_1 \geqslant \varepsilon$ from the state $(\eta_1', \eta_2', \alpha')$ obtained by way of the realization $u^{\circ} = \mu_1^{\circ}\delta$ according to Eqs. (2.1) with u_1° taken in accordance with (2.2). Since the equation $(\eta_1 + \alpha\gamma_1)^2 + (\eta_2 + \alpha\gamma_2)^2 = 1$ along this segment of the trajectory, we infer that the quantity α can be taken from it according to the formula

$$\alpha = -\eta_1\gamma_1 - \eta_2\gamma_2 + \sqrt{1 - (\eta_1\gamma_2 - \eta_2\gamma_1)} \tag{3.1}$$

and we can integrate system (1.1) for $v^{\circ} = 0$ until realization of the equation $\psi_2(t_1) = 0$, i.e. of the boundary with the domains D_3 or D_4, provided that the inequality $w_1(t) > \varepsilon$ is satisfied throughout the time $0 < t \leqslant t_1$. In this case $T_2 = t_1 - T_1$. On the other hand, if the equation $w(t_2) = \varepsilon$ is realized before the equation $\psi_1 = 0$, then from that time on the control v° is realized according to (2.9) and the point (η_1, η_2) moves along the straight line $w_1 = \varepsilon$ until it emerges onto the plane $\psi_2 = 0$ at the instant t_3. In this case $T_2 = t_3 - T_1$. The quantities $t_1 - T_1$ and $t_2 - T_1$ cannot be computed explicitly because of the nonlinearity of the defining equations. The time $t_3 - t_2$ of motion along the straight line $w_1 = \varepsilon$ can be obtained by integrating an equation with separated variables, its explicit form, however, is essentially immaterial.

According to (2.6), (2.8) and Rule 2.10, motion from the domains D_3 and D_4 following the realization of u°, v° occurs along the arc (a_3b_1) in the time $T_3 = 2 \operatorname{arc\,tg} [(\eta_1 + \alpha\gamma_1) / (1 - \eta_2 - \alpha\gamma_2)]$.

Applying theorems on the differentiability of the solution with respect to the initial data, we can show that the function T_3 and its partial derivatives are continuous for all interior points of the domains D_1, D_2, D_3, D_4 lying to the right of the straight line $w_1 = \varepsilon$. The partial derivatives are discontinuous at the points of the plane $\psi_2 = 0$ where the functions T_2 and T_3 are matched, but have limits continuous in the domains D_2, D_3, D_4 as we approach this plane from the domains D_2, D_3, D_4. Such limit points also exist at the remaining boundary points of the domains $D_1, \ldots, D_4$. From now on we shall not distinguish between the derivatives and their interior limits. In fact, the function T_ε is constant at points of the plane $\psi_2 = 0$. Any change in this function is accompanied by a shift out of the plane $\psi_2 = 0$, and the limits of the derivatives can be used to calculate the increment. A lengthy argument can be adduced to demonstrate the possibility of replacing derivatives by their limits.

We also recall that no essentially negative control $u < 0$ is possible on the boundary $R + \alpha = 1$ in the domain D_1, since such a control violates the inequality $R+\alpha \leqslant 1$ and encounter cannot occur by virtue of 2.1 and 2.7.

The same statement is valid for the boundary $r^2 = 1$ of the domain D_2 and for the boundaries $\rho^2 = 1$ of the domain D_3 and $\psi_4 = 0$ of the domain D_4.

Let us begin our investigation of the derivative $(T_\varepsilon)^\cdot$ with the domains D_3 and D_4 where $T_\varepsilon = T_3$. We write the derivative as

$$(T_\varepsilon)^\cdot = (T_3)^\cdot = (1 + w_3^2)^{-1} (1 - \eta_2 - \alpha\gamma_2)^{-2} [\eta_2 (1 - \eta_2 - \alpha\gamma_2) -$$
$$- \eta_1 (\eta_1 + \alpha\gamma_1) + (u_1 + |u_1|) (\eta_1+\alpha\gamma_1) + (v_1 - |v_1|) (\eta_1\gamma_2 - \eta_2\gamma_1 + \gamma_1)]$$

where

$$w_3 = (\eta_1 + \alpha\gamma_1) / (1 - \eta_2 - \alpha\gamma_2)$$

It is easy to show that w_3 is preserved for all $u < 0$, $v > 0$, whether impulsive or finite. The terms of the derivative $(T_\varepsilon)^\cdot$ which contain v_1 are negative for $v_1 < 0$; the terms containing $u_1 > 0$ are positive. Similarly, for $u = \mu_2\delta > 0$ and $v = \nu_2\delta < 0$ we infer that $\Delta T_\varepsilon (\mu_2\delta > 0) > 0$, $\Delta T_\varepsilon (\nu_2\delta < 0) < 0$. On first inspection the derivative $(T_\varepsilon)^\cdot$ appears to be independent of the controls for $u_1 < 0$ and $v_1 > 0$. However, this is not true for impulsive controls $u_1 = \mu_1\delta < 0$ and $v = \nu_1\delta > 0$, since this case entails the variation of a component of the derivative $(T_\varepsilon)^\cdot$ which does not contain u and v.

Denoting the ratios which are preserved for $u = \mu_1\delta < 0$ by

$$w_1 = \eta_1 / (1 - \eta_2), \qquad w_2 = \alpha / (1 - \eta_2)$$

we can rewrite the derivative as

$$(T_\varepsilon)^\cdot = (1 + w_3^2)^{-1} [\eta_2 / (1 - \eta_2) (1 - w_2\gamma_2)^{-1} - w_1 (w_1 + w_2\gamma_1) \times$$
$$\times (1 - w_2\gamma_2)^{-2}]$$

This derivative attains a minimum with respect to $\mu_1 < 0$ for the same value as that which minimizes the quantity η_2. It is clearly equal to the $\mu_1^\bullet$ given by (2.5), since according to (1.5) the quantity η_2' is minimal for the maximum λ. Now let us express $(T_\varepsilon)^\cdot$ in terms of α and the quantities $s_1 = \eta_1 + \alpha\gamma_1$, $s_2 = 1 - \eta_2 - \alpha\gamma_2$ which are preserved for $v = \nu_1\delta > 0$,

$$(T_3)^\cdot = (1 - w_3^2)^{-1} [s_2 - s_1^2 - s_2^2 + \alpha (s_1\gamma_1 - s_2\gamma_2)] s_2^{-2}$$

The expression $s_1\gamma_1 - s_2\gamma_2 = \psi_1 \leqslant 0$ in the domains D_3, D_4, so that $(T_3)^\cdot$ attains its maximum at $\alpha = 0$, i.e. for $\nu_1 = \nu > 0$. We conclude from this that the relations

$$\min_{\mu_1<0} \max_{\nu_1>0} (T_3)^\cdot = \max_{\nu_1>0} \min_{\mu_1<0} (T_3)^\cdot = -1 = (T_3)^\cdot (\mu_1^\circ, \nu_1^\circ)$$

are valid.

We can now draw the final conclusion that the inequalities

$$T (u^\circ, v) \leqslant T_\varepsilon (u^\circ, v^\circ) \leqslant T (u, v^\circ) \tag{3.2}$$

are valid in the domains D_3 and D_4, i.e. that u°, v° realize the saddle point of the game.

In the domain D_2 the function $T_\varepsilon = T_2 + 2\varphi$; in the domain D_1 the function $T_\varepsilon = T_2 + T_1 + 2\varphi$. Noting that $T_\varepsilon(w_1, w_2)$ depends solely on the ratios w_1 and w_2 and denoting the partial derivatives with respect to these variables by T^1 and T^2, we can write $(T_\varepsilon)^{\cdot}$ as

$$(T_\varepsilon)^{\cdot} = T^1 \eta_2 / (1 - \eta_2) - T^1 w_1^2 - T^2 w_1 w_2 + P_1(u) + P_2(v)$$
$$P_1(u) = (1 - \eta_2)^{-2} (u_1 + |u_1|) (T^1\eta_1 + T^2\alpha) \tag{3.3}$$
$$P_2(v) = (1 - \eta_2)^{-2} [v_1 T^1 (\eta_1\gamma_2 - \eta_2\gamma_1 + \gamma_1) + T^2 (\alpha v_1\gamma_2 - \\ - |v_1| (1 - \eta_2))]$$

Let us assume that $T^1 > 0$. This assumption is valid for small α by virtue of continuity since $\alpha = 0$ $T_\varepsilon = 2 \operatorname{arctg} [\eta_2 / (1 - \eta_2)$ for $\alpha = 0$, and since the derivative $T^1 = 1 - \eta_2 > 0$ is positive.

If $T^1 > 0$, the value $(T_\varepsilon)^{\cdot}$ attains its minimum for $\mu_1 < 0$ for the minimal η_2. The minimal η_2 is realized for $u^\circ = \mu_1{}^\circ\delta$ chosen in accordance with (2.3).

We can express this fact in the form

$$T_\varepsilon(\mu_1{}^\circ))^{\cdot} = \min_{\mu_1 < 0} (T_\varepsilon)^{\cdot}$$

Let us introduce the new variables l_1 and l_2 according to the formulas

$$l_1 = \eta_1\gamma_1 + \eta_2\gamma_2, \qquad l_2 = -\eta_1\gamma_2 + \eta_2\gamma_1$$

and prove the validity of an ancillary assumption.

3.1. If the initial point $w_1{}^\bullet > \varepsilon$, $w_2{}^\circ$ lies either in the domain D_2 or in the part of the domain D_1 defined by the inequality

$$-\lambda_1{}^\circ(1 - \eta_2) + \sqrt{\lambda_1{}^{\circ 2}\eta_1{}^2 + [1 - \lambda_1{}^\circ(1 - \eta_2)]^2} - \\ - \lambda_1{}^\circ\eta_1\gamma_1 - [1 - \lambda_1{}^\circ(1 - \eta_2)]\gamma_2 < 0 \tag{3.4}$$

then the variation δT of the time associated with variations of the variables restricted by the conditions $\delta l_2 = 0$, $\delta\alpha = -\delta l_1 > 0$ is positive.

Proof. If the point lies in the domain D_2, the indicated variations leave it in this domain and the impulse $\mu_1{}^\circ$ remains constant (since it depends only on l_2 and $l_1 + \alpha$, and since the latter quantities are preserved). In addition, we can show by direct computation that the variations $\delta l_2{}'$, $\delta l_1{}'$, $\delta\alpha$ obtained after the impulse $\mu_1{}^\circ$ also satisfy the conditions $\delta l_2{}' = 0, -\delta\alpha' = \delta l_1{}' < 0$. From now on we shall omit the primes and define the variations of quantities as their total variations and not their linear approximations. Computing the variation $\delta u_1{}^\circ$ by formula (2.4) for $v^\circ = 0$, we obtain

$$\delta u_1{}^\circ = l_2{}'\delta\alpha / (1 - \eta_2{}' - \alpha'\gamma_2) \leqslant 0 \tag{3.5}$$

since $l_2{}' \leqslant 0$ in the domain D_2. Omitting the primes accompanying the variables, we obtain the variation $\delta l_2{}^{\cdot}$,

$$\delta l_2^{\cdot} = -\delta l_1 + l_2\delta |u_1{}^\circ| + \gamma_1\delta u_1{}^\circ = \delta\alpha (l_1 + \alpha)(l_1 + \alpha - \gamma_2) \times \\ \times (1 - \eta_2 - \alpha\gamma_2)^{-1} > 0 \tag{3.6}$$

Inequality (3.6) is valid by virtue of the fact that all of the factors are positive in the domain D_2 for $\delta\alpha > 0$. The variation of the expression

$$\delta (d\alpha \quad 0 \quad / dl_2) = \delta [\alpha |u_1{}^\circ| / (-l_1 + l_2 |u_1{}^\circ| + u_1{}^\circ\gamma_1)] < 0 \tag{3.7}$$

satisfies inequality (3.7) by virtue of the fact that the positive numerator and negative denominator increase in accordance with (3.5) and (3.6). This means that the function α (l_2, l_{20}, α_0) increases monotonically in α_0 along a trajectory for any l_2, l_{20} until the equation $w_1 = \varepsilon$ is realized. Hence, if the equation $w_1 = \varepsilon$ is not realized on the initial trajectory, the inequality (3.6) is valid at all its points, so that the straight line $l_2 = -\cos\varphi$ (i.e. the straight line (b_1a_3)) , which is the boundary of the domain D_3 or D_4 is attained earlier on the initial trajectory than on the varied trajectory. On the other hand, if the value $w_1=\varepsilon$ is attained at the point l_2° on the initial trajectory, then this value l_2° is attained earlier than on the varied trajectory, and the representing point of the initial trajectory from then on moves along the straight line $w_1 = \varepsilon$, reaching the boundary of the domain D_3 or D_4 earlier than does the point of the initial trajectory. Thus, $\delta T_2 > 0$ in the domain D_2. If the variables lie in the domain D_1, we can show that the variation δT_1 is positive by direct computation. The variation $\delta\alpha' = \delta\alpha\lambda_1^\circ + \alpha\delta\lambda_1^\circ + \delta\alpha\delta\lambda_1^\circ$ also turns out to be positive under condition (3.4). Since the quantity α' is preserved on the trajectory, the following conditions are realized at the instant $T_1 + \delta T_1$ at the boundary of the domain D_2:

$$\delta l_2 (T_1 + \delta T_1) = 0, \qquad \delta\alpha (T_1 + \delta T_1) = -\delta l_1 (T_1 + \delta T_1) > 0$$

Hence, by what we have just proved, $\delta T_1 > 0$ and the variation $\delta T_\varepsilon = \delta T_1 + \delta T_2 > 0$ is positive. Completing our proof, we note that inequality (3.4) assumes the form $\eta_2' - 1 + \sqrt{\eta_1'^2 + \eta_2'^2} - l_1' < 0$ after the realization of $u = \mu_1^\circ\delta$ taken in accordance with (2.2). Since the quantity $\eta_1'^2 + \eta_2'^2$ is preserved along the trajectory and since the quantities η_2' and $-l_1$ diminish, inequality (3.4) is not violated along the trajectory. The proof is now complete.

Writing out the variation δT_ε, we obtain (in accordance with 3.1) the inequality

$$\delta T_\varepsilon = -\delta\alpha (1 - \eta_2)^{-2} [T^1 (\eta_1\gamma_2 - \eta_2\gamma_1 + \gamma_1) + T^2 (\eta_2 + \alpha\gamma_2 - 1)] + O(\delta\alpha) > 0 \tag{3.8}$$

3.2. The following three estimates are valid:

$$P_2 (v) \leqslant 0 \tag{3.9}$$

$$T_2 > 0 \tag{3.10}$$

$$P_1 (u > 0) > 0 \tag{3.11}$$

P r o o f. Estimate (3.9) follows from (3.8); estimate (3.10) follows from (3.9), the fact that $T^1 > 0$, and the estimates

$$\eta_1\gamma_1 + \gamma_2 (1 - \eta_2) > 0, \qquad 1 - \eta_2 - \alpha\gamma_1 > 0$$

which are valid in the domain D_1. Estimate (3.11) follows from (3.8), $T^1 > 0$, and (3.10). If $v < 0$, the estimate (3.9) becomes a strict inequality, $P_2 (v < 0) < 0$. This follows from (3.8), $T_1 > 0$ and (3.10).

3.3. Let us denote the quantity $T^\circ = \lim T_\varepsilon$ as $\varepsilon \to 0$ by T°. This function satisfies estimates (3.2), ..., (3.11). We need not prove this intuitively obvious statement.

Let us introduce the domain D_5 $[\partial T^\circ / dw_1 = T^{\circ 1} > 0]$ formed out of the domain D_5° $[T^{\circ 1} (\alpha = 0) > 0]$ by continuous variation of α, the domain D_4 $[w_1 \geqslant \varepsilon]$, and the domain $D_6 = D_4 \cup D_5$. We can now formulate the following theorem.

T h e o r e m. If $z \in D_6$, the estimates

$$T_\varepsilon (u^\circ, v^\circ) \leqslant T (u, v^\circ) \tag{3.12}$$

$$T(u^\circ, v) < T^\circ \quad (3.13); \quad T_\varepsilon(u^\circ, v^\circ) \geqslant T_\varepsilon(u^\circ, v^*) \qquad (3.13)$$

are valid. Here v^* is a control which together with u° leaves the point in the domain D_1.

Proof. All of the statements of the theorem follow directly from previous statements 3.1, 3.2, 3.3 and from the results of integrating the corresponding minimax and maximin differential inequalities which we shall not write out here. Differential inequalities and the mean-value theorem can be applied to δ-function type controls $v = \nu_2\delta$ or $u = \mu_2\delta$. This yields the inequalities $\Delta T(\nu_2/\mu_1^\circ) < 0$ and $\Delta T(\nu_2^\circ, \mu_2) > 0$. The proof is complete.

The need to verify the inequality $T^{\circ 1} > 0$ leaves something to be desired, as it cannot be verified directly; all we can do is refer to coarse sufficient condition (3.4) which guarantees the estimate $T^{\circ 1} > 0$.

The results of calculations by computer for $\varphi = 0$ appear in Fig. 2. The quantity $T^{\circ 1}$ at the surface $R + \alpha = 1$ vanishes for the first time at the curve $(a_1, 0)$. Moreover, the lines (c_1, d_1), (c_2, d_2) indicate that the control v° is optimal, and

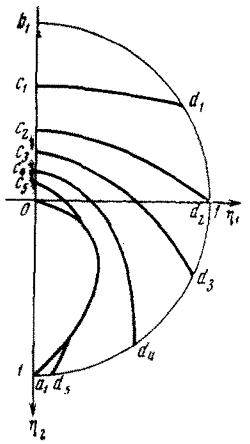

Fig. 2

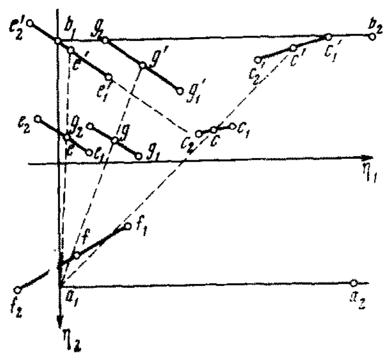

Fig. 3

that $T^{\circ 1} > 0$ even in the domain where inequality (3.4) is violated.

Until now we assumed that the angle φ was acute. If it is obtuse or a right angle, the domain D_3 vanishes and the right end g_1 of the arrow (g_1, g, g_2) becomes the left end. It is therefore sufficient to replace $\sin\varphi$, $\cos\varphi$ by $-\sin\varphi$, $-\cos\varphi$ in all the formulas; no other changes are required. Our analysis of the problem is now complete.

4. We shall cite the results of our analysis of the second example without detailed proofs. The equations of motion in this case are

$$\eta_1^{\cdot} = \eta_2 + \eta_1 |u_1| + v_1\gamma_1, \qquad \eta_2^{\cdot} = \eta_2 |u_1| + u_1 + v_1\gamma_2$$

The restrictions imposed on the controls are the same as in Sect. 1. Once again, we confine our attention to the domain $\eta_1 \geqslant 0$. Figure 3 shows several possible vectors (g_1, g, g_2), (c_1, c, c_2), etc.

It is clear that if one of the arrow ends g_1 or g_2 either lies on the open half-line $(a_1, a_2]$ or intersects the open half-line $(b_1, b_2]$, the impulse $v = \pm\nu\delta$ directed in the appropriate way will take the point outside the attainability domain of zero [3], and pursuit cannot be completed. The vector (f_1, f, f_2), for which the impulse $v^* = -\nu\delta$ directed towards the point f_2 intersects the segment $\eta_1 = 0$, $|\eta_2| \leqslant 1$ and leads to encounter according to the definition of Subsect. 1.1, remains doubtful. Nevertheless, we can show that in this case one of the ends f_1 or f_2 will reach the straight half-lines $(a_1, a_2]$; $(b_1, b_2]$ before encounter for any control "u" and $v = 0$. This enables us to make the following statement.

4.1. If we set v° equal to zero as long as the closed segments with the origin η_1, η_2 and the components $\pm\alpha\gamma_1$, $\pm\alpha\gamma_2$ do not intersect the straight half-line $(b_1, b_2]$ and have no points in common with the half-line $(a_1, a_2]$, and set $v = \pm\nu\delta$ when one of these conditions is violated, then pursuit for an acute angle φ cannot be completed if these conditions are violated at the initial instant. Pursuit cannot be completed for an obtuse angle φ either if these conditions are violated at the initial instant or if at the initial instant $l_2 + \alpha\gamma_2 \geqslant 1$.

Let us cite the formulas for the optimal controls and optimal time without a detailed argument. We assume that all of the inequalities violating the conditions adopted in 4.1 are fulfilled without actually writing them out.

4.2. The angle φ is acute. The control $u^\bullet = \mu_1^\circ\delta$ in the domain D_1 $[\psi_1 = \eta_1 - \alpha\gamma_2 \geqslant 0]$ is of the form

$$\lambda_1^\circ = 2 / (1 - \eta_2 + \alpha \sin \varphi), \qquad \lambda_1^\bullet = \mu / (\mu - |\mu_1^\bullet|) \tag{4.1}$$

If $\lambda_1^\bullet = 1$, the $u_1^\bullet$ taken from the condition $(\lambda_1^\bullet) = 0$ is of the form

$$u_1^\circ = -(^1/_2)(v_1 + |v_1|)\gamma_2 \tag{4.2}$$

In the domain D_2 $(\psi_1 < 0)$ we have

$$\lambda_2^\circ = 2 / (1 - \eta_2 + \eta_1 \operatorname{tg} \varphi), \qquad u_1^\circ = \eta_2 \operatorname{tg} \varphi / (1 - \eta_2)$$

The control $v^\circ = \nu\delta$ everywhere.

The above controls correspond to the saddle point of the game, and the optimal time is given by $\quad T^\circ(u^\circ, v^\circ) = 2(\eta_1 + \alpha\cos\varphi) / (1 - \eta_2 - \alpha \sin\varphi)$

4.3. The angle φ is obtuse. The control u° is given by formulas (4.1) and (4.2). The control $v^\circ = 0$ for $w_1 > \varepsilon$, and must be found from the condition $w_1^\circ = 0$ in the form $v^\circ = \eta_2 / (\gamma_1 - \varepsilon\gamma_2)$ for $w_1 = \varepsilon$. We have a theorem analogous to that of Sect. 3, and the limiting value $T^\circ = \lim T_\varepsilon$ as $\varepsilon \to 0$ is given by the formula

$$T^\circ = \eta_1 / (1 - \eta_2\gamma_1) + \operatorname{ctg} \varphi \ln [(1 - \eta_2 - \alpha\gamma_2) / (1 - \eta_2 + \alpha\gamma_2)$$

Since the derivative $T^{\circ 1}$ is positive and since the restrictions on initial values of the type (3.4) are not essential in this case, the theorem analogous to the theorem of Sect. 3 can be formulated without reference to any not readily verifiable conditions under which the point belongs to a domain analogous to the domain D_6.

BIBLIOGRAPHY

1. Krasovskii, N. N., On the problem of pursuit in the case of linear monotype objects. PMM Vol. 30, №2, 1966.
2. Pozharitskii, G. K., Impulsive tracking in the case of second-order monotype linear objects. PMM Vol. 30, №5, 1966.
3. Markhashov, L. M., Plotnikova, G. V. and Pozharitskii, G. K., Impulse actuated time-optimal operations in second-order linear systems. PMM Vol. 30, №4, 1966.

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